An Improved Measure of Structural Vulnerability

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Abstract: The conventional design provides a structural system with a degree of strength and ductility but it is not easy to quantify the level of robustness. A measure of vulnerability provides a direction to increase the robustness of a structure. The structural vulnerability theory examines the form of a structure to quantify the consequences in relation to initial damage and thus identifies vulnerable failure scenarios. A significant aspect of the theory is the concept of well-formedness which helps to evaluate the structural consequences after damage.

The purpose of this paper is present an improved measure of structural well-formedness and vulnerability. Its novelty lies in taking a better account of the structural supports thus producing more representative vulnerability indices for the identified failure scenarios. First, the structural vulnerability theory, including the original well-formedness measure is briefly introduced to set the context. Second, the modifications to the well-formedness measure are presented. These make use of the largest eigenvalues of the support joints to differentiate the quality of individual support joints to vulnerability. Third, the efficacy of the new measure is demonstrated through numerical examples.

1 Introduction

In structural engineering, the concept of robustness has been explored for over forty years, and design code provisions in some countries include specific requirements to achieve robustness. The conventional design provides a structural system with a degree of strength and ductility. The level of robustness is limited to the safety requirements of structural components and system depending on the specific loads. The location and probability of accidental events like gas explosion, errors in construction or utilization, remains rather uncertain to draw adequate attention to these in the structural design process. But once such events happen, they can cause a large number of casualties and huge economic losses. Such examples include Ronan Point, London in 1968, Alfred P. Murrah Federal Building, Oklahoma City in 1995, and World Trade Centre, New York City in 2001.

If a structure is vulnerable to some kind of loading or damage, it can lead to the collapse of the whole structure or a major part of it. Vulnerability analysis is an approach to assess and quantify the weaknesses of a system. A measure of vulnerability provides a direction to increase the robustness of a structure, which is the ability of a structure to survive the initial damage without affecting the remaining structure. It is related to several aspects of the form of the structure, such as the properties of the members and their connectivity. Such approaches [1-8] are based on examining the stiffness matrix in different ways. These have the potential to identify inherent weaknesses in a structure and a good way to address the uncertainty associated with the response of a structure to low-probability, high-consequence events.

The structural vulnerability theory, developed at Bristol [1-3,5,7] addresses the issue of robustness of structures through an assessment of vulnerability. It examines the form of a structure to quantify the consequences in relation to initial damage and to identify vulnerable failure scenarios. The analysis is independent of external events. A significant aspect of the theory is the concept of well-formedness. It helps to evaluate the structural consequences after damage. Any damage to the structural form leads to the deteriorated stiffness matrices of associated structural members and hence a loss of well-formedness. The analysis leads to vulnerable scenarios but the 'column loss' scenario usually considered in progressive collapse studies remains rather hidden. Because of its vicinity to supports the well-formedness measure requires further considerations which are addressed in this paper.

The paper is organised as follows. The structural vulnerability theory is briefly reviewed to set the context. Then the modifications to the well-formedness measure are identified and described. The new measure is examined through example structures.

2 Structural Vulnerability Theory

Structural vulnerability theory is a theory of form and connectivity [1-3,5,7]. The theory uses the structural properties to define a measure of well-formedness. This measure is used to group members into clusters of increasing size until the whole structure is one large cluster. This process leads to a hierarchical representation of the structure. This hierarchy is systematically unzipped to look for the vulnerable scenarios. The significant aspects of the theory are: (i) the identification of structural system, (ii) the calculation of well-formedness, (iii) the process of clustering for hierarchy and (iv) the process of unzipping for failure scenarios.

2.1 Structural system

A structural system is formed starting from structural members linked through joints. The type of a joint contributes to the ability of a structure to resist damage. A joint between a structural member and the support (such as the ground) will be referred to as 'root joint'. If a member or a joint is removed, the ability of the structure is reduced because of the loss of a load path. A *structural ring* is a minimum structural path which can resist load from any arbitrary direction.

A *structural cluster* is a set of structural rings which are connected with each other. The members within a cluster are tightly connected than the members outside of the cluster. A reference cluster consists of the structural supports, normally the ground. The measures to define a cluster are significant as they can result in different outcomes. The criteria used to form a cluster are the well-formedness of a structural cluster, the minimum damage demand of a structural cluster, the nodal connectivity of a structural cluster and the distance from the reference to the structural cluster. The well-formedness is the most important criterion.

2.2 Well-formedness and damage demand

Well-formedness is a measure to examine the form of structure. It is related to the type of joints, the material of structure and the configuration of members. All of these appear in the stiffness matrix of the structure. Hence well-formedness (q) of a joint is defined in terms of the eigenvalues of the sub-matrix associated with the joint, i.e.

$$q_i = det(\boldsymbol{K}_{ii}) \tag{1}$$

where i denotes the joint number and $\boldsymbol{K}_{ii}$ is the stiffness matrix of the joint i at which the members are connected. The well-formedness of a structure (or a cluster) is obtained as the average well-formedness of all the joints within the structure (or the cluster), i.e.

$$Q = \frac{1}{N}\sum_{i=1}^{N} q_i \tag{2}$$

where N is the total number of joints. The well-formedness measure is an approximation to the determinant of the whole system but the higher the well-formedness of a structure, the better the quality of its form is.

In vulnerability theory, the damage demand is a measure of the effort required to cause a deteriorating event. Although damage can result from widely varying actions, the demand depends upon the properties of the structural member. It is assumed to be directly proportional to the loss of the principal stiffness caused by a deteriorating event. This enables the analysis to remain independent of the nature of hazards.

2.3 The clustering and unzipping process

2.3.1 Clustering

Structural rings and clusters with better quality of form are identified through the clustering process. The clustering process begins from the elementary level (i.e. a leaf cluster) and recruits neighbouring members and joints to form a higher level of cluster. The members within a cluster are better connected as compared to the rest of the structure. Clusters at each level of a hierarchy contain information about the members and joints at the lower level. The clustering process leads to a hierarchical representation of the structure which is well-suited to guide a search of potential failure scenarios of interest.

2.3.2 Unzipping and failure scenarios

Unzipping introduces a deteriorating event in selected clusters starting from the top of hierarchy. The search process is continued until a cluster can be damaged by one or more deteriorating events. The search stops when the structure becomes a mechanism. After potential failure scenarios have been found, the consequences of each scenario are assessed through a measure of separateness and vulnerability (given subsequently). The detailed descriptions and algorithms for the clustering process and the unzipping process are given by Yu [2] and Pinto [7]. Important failure scenarios include: maximum failure scenario – one which leads to the most damage with minimum damage, total failure scenario – one where the whole structure fails with least damage, minimum failure scenario – the easiest way to cause damage to a structure irrespective of its disproportionateness.

2.3.3 Separateness

Deterioration events, consisting of a pin or a cut in one or more members, generally lead to a different degree of damage to the structure, e.g. the complete failure, partial collapse or damage to a few members, etc. To represent the scale of damage, the measures of separateness and vulnerability have been defined.

Separateness (γ) indicates how disconnected the clusters are and it is obtained as,

$$\gamma = \frac{Q-Q'}{Q} \tag{3}$$

where Q is the well-formedness of the intact structure and Q' is the well-formedness of the deteriorated structure. The separateness corresponds to a failure scenario and it ranges between 0 and 1.

2.3.4 Vulnerability index

The vulnerability index (ξ) is an indicator of the scale of consequences in relation to the damage and has a range $0 < \xi < \infty$. It is calculated as

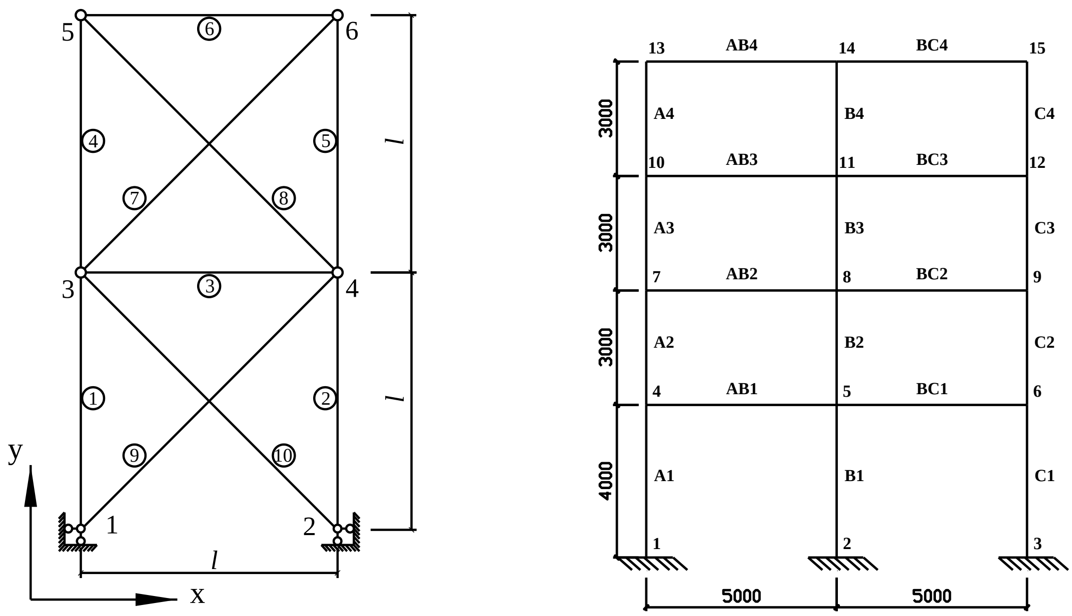
$$\xi = \frac{\gamma}{D_r} \tag{4}$$

where D_r is the relative damage demand. Maximal failure scenario has the highest vulnerability index and minimal failure scenario corresponds to the least effort required to cause damage.

3 An Improved Measure of Vulnerability

3.1 Motivation for the modification

We consider two example structures [9] to examine their vulnerability and the potential improvements. The layout of Example 1, a truss, is given in Figure 1(a). All the members have the same cross-sectional area and Young's modulus. Example 2 is a two-bay four storey concrete frame as shown in Figure 1(b). The cross-section for beams is $200 \times 300 \text{mm}^2$ and that for the ground and upper storey columns are $500 \times 500 \text{mm}^2$ and $450 \times 450 \text{mm}^2$, respectively. The structure has moment-resisting joints.



(a) (b)

Figure 1: The layout of (a) Example 1 and (b) Example 2

The measures of well-formedness and separateness were used to compare the quality of form of each structure after some damage. The deteriorating events were the removal of members, especially those directly connected to the ground.

For Example 1, these are member ① (Case 1.1), member ④ (Case 1.2) and member ⑩ (Case 1.3). Table 1 (second column) shows well-formedness, separateness, relative damage demand and vulnerability index for the three cases. The results show that the failure scenario in Case 1.2 is the most vulnerable based on structural form. In the presence of gravity loads, it can be argued that the loss of the ground storey vertical members should cause higher consequences to the structure than the loss of upper storey members unless impact due to debris becomes a governing factor.

Table 1: Results of Example 1 (a) before modification and (b) after modification

	Before modification			*After modification*		
Case No.	1.1	1.2	1.3	1.1	1.2	1.3
Well-formedness $Q'\left(\frac{EA}{l}\right)^2$	1.884	1.717	1.908	4.590	6.053	5.595
Separateness	0.154	0.229	0.143	0.255	0.078	0.147
Vulnerability Index	1.363	2.027	1.788	2.257	0.690	1.838
System integrity distance metric [4] (lower value indicates higher vulnerability)				0.018	0.051	0.049

The structure in Example 2 was examined with the removal of one member at a time, each representing a failure scenario, and the vulnerability results are summarised in Table 2. The results show that the removal of first storey beam causes the largest vulnerability index. Amongst columns, again the first storey (Columns A2 and B2) is the most critical. Considering gravity loads, the significance of ground storey columns cannot be underestimated.

Table 2: Comparison of different failure scenarios for Example 2

Removed Member	AB1	AB2	AB3	AB4	A1	A2
Well-formedness Q' ($\times 10^{26}$)	3.736	3.750	3.750	4.175	4.350	3.791
Separateness	0.137	0.134	0.134	0.036	-0.005	0.124
Relative Damage Demand	0.004	0.004	0.004	0.004	0.052	0.069
Vulnerability Index	32.67	31.88	31.88	8.55	-0.087	1.80
Removed Member	**A3**	**A4**	**B1**	**B2**	**B3**	**B4**
Well-formedness Q' ($\times 10^{26}$)	3.786	3.983	4.109	3.352	3.351	3.683
Separateness	0.126	0.080	0.051	0.226	0.226	0.150
Relative Damage Demand	0.069	0.069	0.052	0.069	0.069	0.069
Vulnerability Index	1.82	1.16	0.98	3.26	3.27	2.16

These issues appear to relate to the supports and in particular how a support is considered when a member directly connected to it is lost. This is addressed in this paper through modifications to the well-formedness calculations.

3.2 The increased significance of the root joints

The stiffness sub-matrices of joints are extracted from the stiffness matrix of the whole structure. The well-formedness of joints is obtained from the product of eigenvalues of the corresponding stiffness sub-matrices. In a typical structure, most joints are between structural members and only a few members are directly connected to the root joints i.e. the supports. The global stiffness matrix contains the entries related to structural members associated with the root joints but the influence of the ground (reference cluster) is not included. Therefore, the well-formedness of a root joint turns out to be relatively small in spite of infinitely large stiffness of the ground in most cases. Thus, the calculation procedure relating to the root joints does not adequately reflect the significance of the ground.

If there is only one member connected to the ground, the determinant of the matrix associated with the root joint is zero (e.g. the root joint 1 in Case 1.1 of Example 1) which is contrary to its contribution to the structural rings, and even if more than one members are connected to the ground, the determinant still happens to be small. Since the root joint is firmly supported by the reference cluster, the root joint should have a higher well-formedness. If that is so, it would result in a larger proportion in the well-formedness of the structure. This in turn would result in the importance of the member directly connected to the ground. Thus, the well-formedness measure used previously can be improved to include the influence of root joints.

But the questions such as ‘how much well-formedness comes from root joints?’ or ‘what should be the rule for increasing well-formedness for root joint?’ arise. If the values are extremely large, it is difficult to tell which member is the most vulnerable apart from ground members. The increase in well-formedness should be comparable and of the same order as the other values. Therefore, a solution to appropriately increase the well-formedness is needed.

A possible solution is to increase the well-formedness by substituting the determinant of stiffness matrix of root joint with the sum of determinants of all the joints which is noted as the cumulative well-formedness. The cumulative well-formedness of the joints is high enough to represent well-formedness of a root joint and it is also not far from the well-formedness of other joints because the cumulative well-formedness of the joints is obtained based on their determinants. Thus, the significance of root joints can be emphasised but this is only a part of the solution. All the root joints are not the same and they must represent the type of joint or incident members. However, before the damage, the root joints are more affected by the ground as compared to the structural members. Thus, the well-formedness of each root joint before damage can be represented by the cumulative well-formedness of the joints.

3.3 The contribution of members connected to the ground

Removing the ground member causes more damage to the structure but if a root joint has two or more members, their contribution to the structure may be different because of their properties or orientations. So, a decision on the reduction of well-formedness of the root joints after damage to any one of the incident members becomes significant. For example, the stiffness matrices for the vertical and diagonal ground members at a root joint in Example 1 will lead to quite different determinants. Therefore, the contributions of associated members to the well-formedness of root joints need to be defined.

Nafday [4] used the smallest singular value for vulnerability analysis. The smallest singular value is the distance of stiffness matrix from the nearest singularity. The smallest singular value corresponding to stiffness matrix of a root joint might be zero and hence it cannot be used for well-formedness. In the case of a normal matrix, the singular values are simply the absolute values of the eigenvalues. For a positive definite matrix, the eigenvalues are positive. Hence the largest eigenvalue is the largest singular value. A comparison of the largest eigenvalue before and after an external event can be an indicator of structural vulnerability. The largest eigenvalue from the intact stiffness matrix represents the intact root joint that is connected to all the original ground members. The largest deteriorated eigenvalue is obtained from the deteriorated stiffness matrix after losing a ground member. Therefore, the influence of deteriorated ground member on the remaining structure could be taken into account by the ratio of the largest eigenvalue from deteriorated stiffness matrix to the largest eigenvalue from the intact stiffness matrix. This ratio will be referred to as the *deteriorating ratio* ($\frac{\lambda_l^{'}}{\lambda_l}$), where λ_l and $\lambda_l^{'}$ are the largest eigenvalues of stiffness matrices for the intact root joint and the deteriorated root joint, respectively. The ratio demonstrates the significance of a damaged ground member on the quality of the adjacent root joint. The smaller the ratio, the larger consequence the loss of a member causes to the structure.

3.4 The algorithm for the modified well-formedness

There are three steps to calculate the modified well-formedness for structural vulnerability:

Step 1 Calculate the cumulative well-formedness of all the joints before any damage to the structure and assign this value to the root joints, i.e.

$$q_{root} = \sum_{i=1}^{N} q_i \tag{6}$$

where q_{root} is the intact well-formedness for a root joint, *N* is the total number of joints and q_i is the well-formedness of joint *i*.

Step 2 Extract the largest eigenvalues (λ_l and $\lambda_l^{'}$) from the stiffness matrices of the intact and the deteriorated root joints and determine the deteriorating ratio.

Step 3 Obtain the updated well-formedness ($q_{root}^{'}$) of root joints affected by the removed ground members as follows:

$$q_{root}^{'} = q_{root}\frac{\lambda_l^{'}}{\lambda_l} \tag{7}$$

4 Results using the Proposed Measure

4.1 Example 1

The proposed modification to well-formedness is applied to Example 1 described earlier in Section 3.1. The results before and after modifications are summarised in Table 1. The proposed modification shows that the deteriorated structure in Case 1.1 has the smallest well-formedness and the highest vulnerability index, so it is the most susceptible to damage. With the increased determinants of the root joints, the significance of ground member stands out. Case 1.1 is followed by Case 1.3 in terms of vulnerability index. In the latter case also, a ground member is removed but this corresponds to the loss of a diagonal member. The removal of top storey members causes smaller consequences to the structure than the removal of ground members.

The results are also compared against system integrity distance metric [4] which examines the shortest distance from the stiffness matrix of a structure to singularity by condition number. The safety metrics for Cases 1.1, 1.2 and 1.3 are 0.018, 0.051 and 0.049. The lower the metric, the higher the vulnerability. Therefore Case 1.1 is the highest vulnerable scenario, and Case 1.2 is the least. Thus, the modification to the well-formedness of root joints improves the vulnerability analysis.

4.2 Example 2

Example 2 from Section 3.1 is also repeated to test the modified vulnerability analysis. The well-formedness and separateness of failure scenarios, obtained from the modified measure are presented in Table 3.

Table 3: Comparison of different failure scenarios for Example 2 (after modification)

Removed Member	AB1	AB2	AB3	AB4	A1	A2
Well-formedness Q' ($\times 10^{26}$)	1.672	1.673	1.673	1.716	1.271	1.677
Separateness	0.034	0.034	0.034	0.009	0.266	0.031
Relative Damage Demand	0.004	0.004	0.004	0.004	0.052	0.069
Vulnerability Index	8.17	7.98	7.98	2.14	5.11	0.45
System integrity distance metric ($\times 10^{-4}$, [4])	0.603	0.673	0.687	0.694	0.368	0.453
Removed Member	**A3**	**A4**	**B1**	**B2**	**B3**	**B4**
Well-formedness Q' ($\times 10^{26}$)	1.677	1.696	1.249	1.633	1.633	1.666
Separateness	0.031	0.020	0.279	0.057	0.057	0.037
Relative Damage Demand	0.069	0.069	0.052	0.069	0.069	0.069
Vulnerability Index	0.45	0.29	5.36	0.82	0.82	0.54
System integrity distance metric ($\times 10^{-4}$, [4])	0.545	0.568	0.677	0.752	0.765	0.790

The results show that after the modification to the well-formedness measure, the scale of damage caused by the removal of structural members is getting lower from the bottom to the top. Since the middle column has more communication channels with the structure than the peripheral columns, the removal of middle column causes a larger damage to the structure. Amongst all members, the ground middle column B1 causes the most separateness. Along the periphery, the ground columns also have higher separateness than the columns above. The system integrity distance metric corresponding to the failure scenario of ground columns also produces lower values as compared to the upper columns. The negative separateness encountered for some cases in Table 2 is no longer observed after the proposed modification.

The separateness of beams becomes lower for the upper storeys which is consistent with the results obtained using distance metric. For the beams in the first and second storeys, they have the same vulnerability parameters because the stiffness matrices of the joints are associated with the members with the same structural properties. If debris loading was to be considered, these are likely to be different.

4.3 Example 3

The modification to well-formedness has the potential to affect the structural hierarchy and hence the failure scenarios due to the involvement of the ground. The failure scenarios are produced during the unzipping process and the question whether these are affected by the modification needs further analysis. A truss structure (Figure 3) adapted from [7], is used to study the differences in vulnerable failure scenarios, if any, after the proposed modification.

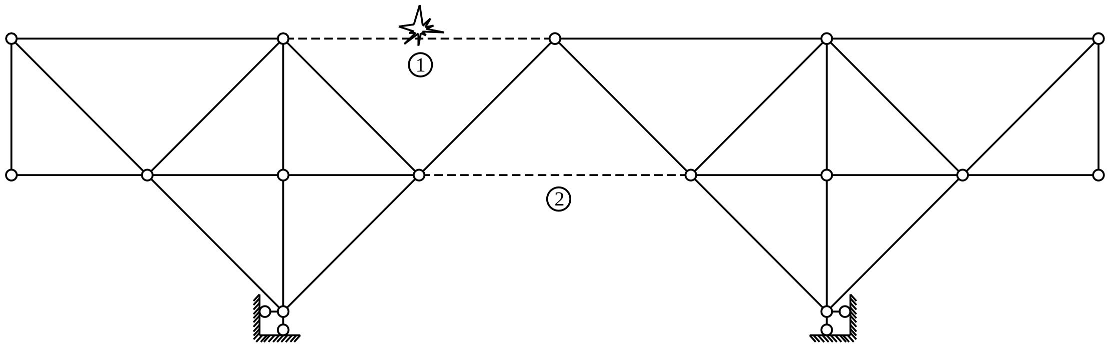


Figure 2: The maximum failure scenario from vulnerability theory (before and after the modification)

The interesting failure scenarios including the maximum failure scenario (shown by dashed lines in Figure 2) are the same from the modified vulnerability analysis and from the original vulnerability analysis. This finding can be explained based on the processes of clustering and unzipping. The clustering process starts from the basic cluster by checking its well-formedness, minimum damage demand, and nodal connectivity, etc., and the last cluster to join in the structure is the ground. Therefore, the ground does not interfere with the clustering process, at least in this example, and the same is the case for the unzipping process. One of the unzipping criteria is to unzip the cluster that directly connects to the ground, and this makes the selected cluster avoid being affected by the modification of the well-formedness on ground members. Therefore, the modification does not change the failure scenario but some effect is noted in the vulnerability index, as was the case for the other two examples.

5 Conclusions

a) The structural vulnerability theory evaluates the quality of the structural form independent of external loads. One significant aspect of this theory is the measure of well-formedness. This relates to the eigenvalues of the stiffness matrices associated with the joints.
b) The original analysis quantifies the vulnerability of structures but the members adjacent to the supports were found to cause smaller consequences when damage occurs on them. This relates to the treatment of supports in the analysis.
c) The summation of the well-formedness of the joints and the largest eigenvalues form a good basis for the modification to the well-formedness associated with the root joints. The cumulative well-formedness of the joints is used to increase the well-formedness of root joints and the largest eigenvalue is used to differentiate the quality of individual root joints for vulnerability analysis. Analyses of example structures show that the proposed modification is able to address the issues related to the loss of ground members.
d) The outcome for structural vulnerability has been verified and the interesting failure scenarios remain unaffected by the modification. The examples presented show that the modified vulnerability theory provides a practical tool with solid basis for comparing the quality of structural forms.

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